6.631: Optics and Optical Electronics Formulæ

1 Constants

$$c = 2.9979 \times 10^8 \text{ m/s}$$
$$\epsilon_0 = 8.8542 \times 10^{-12} \text{ F/m}$$
$$\mu_0 = 1.2566 \times 10^{-6} \text{ H/m}$$
$$\eta_0 = 376.73\ \Omega$$
$$q = 1.6022 \times 10^{-19} \text{ C}$$

2 Mathematics

2.1 Vector Calculus

$$\int_V \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{a}$$
$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\ell$$
$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$
$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0,\ \nabla \times (\nabla \Phi) = 0$$

2.2 Fourier Transform Theorems

$$f(t) \equiv \int_{-\infty}^{\infty} F(\omega) e^{i\omega t} d\omega$$
$$F(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t} dt$$

$f(at)$	$F(\omega/a)/\lvert a \rvert$
$f^*(t)$	$F^*(-\omega)$
$F(t)$	$f(-\omega)/2\pi$
$f(t-\tau)$	$F(\omega)e^{-i\tau\omega}$
$\frac{d^n f(t)}{dt^n}$	$(i\omega)^n F(\omega)$
$e^{-\alpha t^2}$	$\sqrt{\frac{\pi}{\alpha}} e^{-\omega^2/4\alpha}$
$e^{-\alpha t} U(t)$	$\frac{1}{\alpha + i\omega}$

3 Statics

3.1 Maxwell's Equations

3.1.1 Differential Form

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$
$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$
$$\nabla \cdot \mathbf{D} = \rho$$
$$\nabla \cdot \mathbf{B} = 0$$
$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon \mathbf{E}^2 + \frac{1}{2}\mu \mathbf{H}^2\right) + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mu_0 \mathbf{M}}{\partial t} + \mathbf{E} \cdot \mathbf{J}$$

3.1.2 Boundary Condition Form

$$\hat{\mathbf{n}} \cdot (\mathbf{D}_{\text{out}} - \mathbf{D}_{\text{in}}) = \sigma_s$$
$$\hat{\mathbf{n}} \cdot (\mathbf{B}_{\text{out}} - \mathbf{B}_{\text{in}}) = 0$$
$$\hat{\mathbf{n}} \times (\mathbf{E}_{\text{out}} - \mathbf{E}_{\text{in}}) = 0$$
$$\hat{\mathbf{n}} \times (\mathbf{H}_{\text{out}} - \mathbf{H}_{\text{in}}) = \mathbf{K}$$

3.2 Lorenz Force Equation

$$\mathbf{f} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

3.3 Dielectric

$$\mathbf{P} \equiv Nq\mathbf{d}$$
$$\mathbf{D} \equiv \epsilon_0 \mathbf{E} + \mathbf{P}$$
$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$
$$\epsilon \equiv \epsilon_0 (1 + \chi_e)$$

3.4 Reference Formulæ

$$\mathbf{E}_{\text{point}} = \frac{q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}},\ \mathbf{E}_{\text{line}} = \frac{\lambda}{2\pi\epsilon_0 r}\hat{\mathbf{r}}$$
$$\Phi_{\text{point}} = \frac{q}{4\pi\epsilon_0 r},\ \Phi_{\text{line}} = \frac{-\lambda}{2\pi\epsilon_0 \ln r}$$

4 Time Harmonic

4.1 Maxwell's Equations

$$\mathbf{A}(\mathbf{r}, t) \equiv \Re\left[\mathbf{A}(\mathbf{r}) e^{-i\omega t}\right]$$
$$\nabla \times \mathbf{E} = i\omega\mu(\omega)\mathbf{H}$$
$$\nabla \times \mathbf{H} = -i\omega\epsilon(\omega)\mathbf{E} + \mathbf{J}$$
$$\langle \mathbf{S} \rangle = \frac{1}{2}(\mathbf{E} \times \mathbf{H}^*)$$

4.2 Wave Equation

$$\nabla^2 \mathbf{E} + \omega^2 \mu\epsilon \mathbf{E} = 0$$
$$k^2 = \omega^2 \mu\epsilon = n^2 \omega^2$$
$$\mathbf{H} = \sqrt{\frac{\epsilon}{\mu}}(\hat{\mathbf{k}} \times \mathbf{E}) = \frac{1}{\eta}(\hat{\mathbf{k}} \times \mathbf{E})$$

4.3 Vector Potential

$$\mu_0 \mathbf{H} = \nabla \times \mathbf{A}$$
$$\mathbf{E} = -\frac{\partial A}{\partial t} - \nabla\Phi$$
$$\nabla \cdot \mathbf{A} + \mu_0 \epsilon \frac{\partial \Phi}{\partial t} = 0$$

4.4 Wave Impedance

TE	**TM**
$\Gamma(z) = \frac{E_-}{E_+}e^{2ik_zz}$	$\Gamma(z) = -\frac{H_-}{H_+}e^{2ik_zz}$
$Z_0 = \eta/\cos\theta$	$Z_0 = \eta\cos\theta$
$Z(z) = -\frac{E_y}{H_x}$	$Z(z) = \frac{E_x}{H_y}$

4.4.1 Common

$$\frac{Z(z)}{Z_0} = \frac{1+\Gamma(z)}{1-\Gamma(z)}$$

$$\Gamma(z) = \frac{Z(z)-Z_0}{Z(z)+Z_0}$$

$$T(z) = \frac{2Z(z)}{Z(z)+Z_0}$$

$$Z(z) = Z_0\frac{Z(0)-jZ_0\tan k_zz}{Z_0-jZ(0)\tan k_zz}$$

4.5 Propagation Matrices

$$p_{(\ell+1)\ell} = \frac{\mu_{\ell+1}k_z^{(\ell)}}{\mu_\ell k_z^{(\ell+1)}}$$

$$\Gamma_{(\ell+1)\ell} = \frac{1-p_{(\ell+1)\ell}}{1+p_{(\ell+1)\ell}}$$

$$\mathbf{V}_{(\ell+1)\ell} = \frac{1}{2}(1+p_{(\ell+1)\ell})\begin{bmatrix} e^{-ik_zd} & \Gamma_{(\ell+1)\ell}e^{-ik_zd} \\ \Gamma_{(\ell+1)\ell}e^{ik_zd} & e^{ik_zd} \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ T \end{bmatrix} = \mathbf{V}_{t0}\begin{bmatrix} R \\ 1 \end{bmatrix}$$

$$r = -\frac{V_{12}}{V_{11}},\ t = -\frac{V_{12}V_{21}}{V_{11}} + V_{22}$$

4.6 Scattering Matrices

$$\mathbf{S} = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}$$

$$\mathbf{S}^t = \mathbf{S}\ \text{(reciprocity)}$$

$$\mathbf{S}^\dagger = \mathbf{S}^{-1}\ \text{(power conservation)}$$

$$|S_{11}|^2 + |S_{21}|^2 = 1$$

$$|S_{22}|^2 + |S_{12}|^2 = 1$$

$$S_{11}^*S_{12} + S_{21}^*S_{22} = 0$$

$$\mathbf{S}^* = \mathbf{S}^{-1}\ \text{(time reversal)}$$

4.7 Fabry-Perot Resonator

$$\mathbf{S} = \frac{1}{1-r_1r_2e^{-j\delta}}\begin{bmatrix} -(r_1-r_2e^{-j\delta}) & -t_1t_2e^{-j\delta/2} \\ -t_1t_2e^{-j\delta/2} & -(r_2-r_1e^{-j\delta} \end{bmatrix}$$

$$\ell(\omega n/c)\cos\theta \equiv \delta/2$$

$$\frac{|b_2|^2}{|a_1|^2} = \frac{(1-R)^2}{(1-R)^2+4R\sin^2(\delta/2)}$$

$$\Delta f = \frac{c}{2n\ell\cos\theta}$$

$$|\Delta\lambda| \approx \lambda\frac{\Delta f}{f} = \frac{\lambda^2}{2n\ell}\cos\theta$$

$$\delta f_{1/2} = \frac{(1-R)c}{2\pi\sqrt{R}n\ell\cos\theta}$$

5 Fourier Optics

5.1 Paraxial Wave Equation

$$k_z \approx k - \frac{k_x^2+k_y^2}{2k}$$

$$\nabla_{\mathrm{T}}^2u - 2jk\frac{\partial u}{\partial z} = 0$$

$$\Psi(x,y,z) = u(x,y,z)\exp(-jkz)$$

5.2 Fresnel Diffraction ($z \gg x$)

$$h(x,y,z) = \frac{j}{\lambda z}e^{-jk[(x^2+y^2)/2z]}$$

$$H(k_x,k_y,z) = \frac{1}{(2\pi)^2}e^{-j[(k_x^2+k_y^2)/2k]z}$$

$$u(x,y,z) = h(x,y,z)\otimes u_0(x,y,z)$$

$$u = \frac{j}{\lambda z}\int_{-\infty}^{\infty}dx_0\int_{-\infty}^{\infty}dy_0\ u_0(x_0,y_0)e^{-j(k/2z)[(x-x_0)^2+(y-y_0)^2]}$$

$$u_{f-f}(x,y) = j\frac{(2\pi)^2}{\lambda f}U_0(kx/f,ky/f)$$

5.3 Fraunhöfer Diffraction ($d \ll \sqrt{z/k}$)

$$u(x,y,z) = j\frac{(2\pi)^2}{\lambda z}\exp\left[-j\frac{k(x^2+y^2)}{2z}\right]U_0\left(\frac{kx}{z},\frac{ky}{z}\right)$$

$$u_{\text{slit}} = \frac{jd_xd_y}{\lambda z}\exp\left[-\frac{jk(x^2+y^2)}{2z}\right]\frac{\sin(kd_xx/2z)\sin(kd_yy/2z)}{(kd_xx/2z)(kd_yy/2z)}$$

$$A(x,z) = \frac{\sin(Nk\Lambda x/2z)}{\sin(k\Lambda x/2z)}$$

5.4 Gaussian Optics

5.4.1 Fundamental

$$u_{00}(x,y,z) = j\sqrt{\frac{kb}{\pi}}\left(\frac{1}{z+jb}\right)\exp\left[-jk\frac{x^2+y^2}{2(z+jb)}\right]$$

$$= \sqrt{\frac{2}{\pi w^2}}\exp(j\phi)\exp\left[-\frac{x^2+y^2}{w^2}\right]\exp\left[-\frac{jk}{2R}(x^2+y^2)\right]$$

$$w^2(z) = \frac{2b}{k}\left(1+\frac{z^2}{b^2}\right)$$

$$\frac{1}{R(z)} = \frac{z}{z^2+b^2}$$

$$\phi = \tan^{-1}\frac{z}{b}$$

$$w_0 = \sqrt{2b/k}$$

$$b = z_0 = kw_0^2/2$$

$$d_2 = \frac{f^2(d_1-f)}{(d_1-f)^2+(\pi w_1^2/\lambda)^2} + f$$

$$w_2 = \left[\frac{1}{w_1^2}\left(1-\frac{d_1}{f}\right)^2 + \frac{1}{f^2}\left(\frac{\pi w_1}{\lambda}\right)^2\right]^{-1/2}$$

5.4.2 Resonant Modes

$$4b^2 = \frac{R_1^2 R_2^2}{4} \frac{1 - \left[(2d/R_1R_2)(d - R_1 - R_2) + 1\right]^2}{\left[d - (R_1 + R_2)/2\right]^2}$$

$$0 \leq \left(1 - \frac{d}{R_1}\right)\left(1 - \frac{d}{R_2}\right) \leq 1$$

5.4.3 q-Parameter

$$\frac{1}{q} \equiv \frac{1}{z + jb} = \frac{1}{R} - j\frac{\lambda}{\pi w^2}$$

$$q' = q + d,\ \frac{1}{q'} = \frac{1}{q} - \frac{1}{f}$$

$$q' = \frac{Aq + B}{Cq + D}$$

5.5 $ABCD$ Matrices

$$\begin{bmatrix} r_2 \\ r_2' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}$$

free space: $\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$

thin lens: $\begin{bmatrix} 1 & 0 \\ -1/f & 1 \end{bmatrix}$

mirror: $\begin{bmatrix} 1 & 0 \\ -2/R & 1 \end{bmatrix}$

dielectric interface: $\begin{bmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{bmatrix}$

spherical dielectric: $\begin{bmatrix} 1 & 0 \\ \frac{n_2 - n_1}{n_2 R} & \frac{n_1}{n_2} \end{bmatrix}$

$$\frac{1}{\ell_1} + \frac{1}{\ell_2} = \frac{1}{f},\ M = \left|\frac{\ell_2}{\ell_1}\right|$$

$$q_{\text{mode}} = \frac{A - D}{2C} \pm \sqrt{\left(\frac{A + D}{2C}\right)^2 - \frac{1}{C^2}}$$

$$b' = \frac{b_0}{C^2 b_0^2 + D^2} \text{ (waist-to-waist)}$$

$$b_0 = \sqrt{\frac{1 - D^2}{C^2}} \text{ (self-consistent)}$$

5.6 Guided Modes

5.6.1 Slab Waveguide

$$\frac{\alpha_x}{k_x} = \tan k_x d = \begin{cases} \sqrt{\frac{\omega^2 \mu_0 (\epsilon_i - \epsilon)}{k_x^2} - 1} & \text{(TE)} \\ \frac{\epsilon_i}{\epsilon}\sqrt{\frac{\omega^2 \mu_0 (\epsilon_i - \epsilon)}{k_x^2} - 1} & \text{(TM)} \end{cases}$$

5.6.2 LP Fiber Modes

$$V = k_0 a \sqrt{n_1^2 - n_2^2}$$

$$J_{l-1}(V) = 0$$

5.7 Perturbation Theory

$$\epsilon(r) = \epsilon(r) + \epsilon'(r)$$

$$\Psi(x, y, z) = \sum_n c_n(z)\mathcal{E}_n(x, y)e^{-j\beta_n z}$$

$$\frac{\partial c_n}{\partial z} e^{-j\beta_m z} = \sum_n c_n e^{-j\beta_n z}\kappa_{mn}$$

$$\kappa_{mn} = \frac{-j\omega}{4}\int \mathcal{E}_m^* \epsilon' \mathcal{E}_n d\mathbf{A}$$

5.8 Coupled Modes

$$\kappa_{mn} = -p\kappa_{nm}^*, \quad \kappa_{mn} = j\kappa$$

5.8.1 Two Mode

$$\begin{bmatrix} \beta - (\bar\beta + \delta) & \kappa \\ \kappa & \beta - (\bar\beta - \delta) \end{bmatrix} \mathbf{a} = \mathbf{0}$$

$$\beta^2 - \delta^2 - \kappa^2 - 2\beta\bar\beta + \bar\beta^2 = 0$$

$$\beta = \bar\beta \pm \sqrt{\delta^2 + \kappa^2}$$

$$\mathbf{a}_- = \begin{bmatrix} -\frac{\delta + \sqrt{\delta^2 + \kappa^2}}{\kappa} \\ 1 \end{bmatrix}, \quad \mathbf{a}_+ = \begin{bmatrix} \frac{\delta - \sqrt{\delta^2 + \kappa^2}}{\kappa} \\ -1 \end{bmatrix}$$

5.8.2 Three Mode

$$\begin{bmatrix} \beta - (\bar\beta + \delta) & \kappa & 0 \\ \kappa & \beta - \bar\beta & \kappa \\ 0 & \kappa & \beta - (\bar\beta - \delta) \end{bmatrix} \mathbf{a} = \mathbf{0}$$

$$\beta = \left\{\bar\beta, \bar\beta \pm \sqrt{\delta^2 + 2\kappa^2}\right\}$$

$$\mathbf{a}_0 \approx \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{a}_- \approx \begin{bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{bmatrix}, \quad \mathbf{a}_+ \approx \begin{bmatrix} 1 \\ \sqrt{2} \\ 1 \end{bmatrix}$$

6 Ultrafast Optics

$$\tau_{\text{out}} = \tau\sqrt{1 + \left(\frac{\tau_c}{\tau}\right)^4}$$

$$\tau_c \equiv \left.\sqrt{\ell\frac{\partial^2\beta}{\partial\omega^2}}\right|_{\omega=\omega_0}$$