6.431: Probabilistic Systems Analysis

Jonathan Birge

1 Set Theory

$$\left(\bigcup_n S_n\right)^C = \bigcap_n S_n^C$$

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2 Probability Law

2.1 Axioms

$$\mathbf{P}(A) \geq 0, \quad \forall A \in \Omega$$
$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) \leftarrow A \cap B = \varnothing$$
$$\mathbf{P}(\Omega) = 1$$

2.2 Properties

$$\mathbf{P}(A \cup B) = \mathbf{P}(A) + \mathbf{P}(B) - \mathbf{P}(A \cap B)$$
$$\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(A^c \cap B) + \mathbf{P}(A^c \cap B^c \cap C)$$

$$\mathbf{P}(A \cup B \cup C) = \mathbf{P}(A) + \mathbf{P}(B) + \mathbf{P}(C) - \mathbf{P}(A \cap B) - \mathbf{P}(A \cap C) - \mathbf{P}(B \cap C) + \mathbf{P}(A \cap B \cap C)$$

$$\mathbf{P}(A_1 \cup A_2 \cup \cdots \cup A_n) = \left(\sum_i \mathbf{P}(A_i)\right) - \left(\sum_{1\leq i<j\leq n} \mathbf{P}(A_i \cap A_j)\right) + \left(\sum_{1\leq i<j<k\leq n} \mathbf{P}(A_i \cap A_j \cap A_k)\right) - \cdots + \left((-1)^{n-1}\mathbf{P}(A_1 \cap A_2 \cap \cdots \cap A_n)\right)$$

3 Conditional Probability

3.1 Definition

$$\mathbf{P}(A|B) = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(B)}$$

3.2 Total Probability

$$A_i \cap A_k = \varnothing, \quad i \neq k \rightarrow$$
$$\mathbf{P}(B) = \sum_i \mathbf{P}(A_i \cap B) = \sum_i \mathbf{P}(B|A_i)\mathbf{P}(A_i)$$

3.3 Bayes' Rule

$$\mathbf{P}(A_i|B) = \frac{\mathbf{P}(B|A_i)\mathbf{P}(A_i)}{\mathbf{P}(B)} = \frac{\mathbf{P}(B|A_i)\mathbf{P}(A_i)}{\sum_i \mathbf{P}(B|A_i)\mathbf{P}(A_i)}$$

4 Independence

$$\mathbf{P}\left(\bigcap_{i\in S} A_i\right) = \prod_{i\in S}\mathbf{P}(A_i), \quad \forall S \subseteq \{1, 2, \ldots, n\}$$

$$\mathbf{P}(A \cap B) = \mathbf{P}(A)\mathbf{P}(B) \rightarrow$$
$$\mathbf{P}(A|B) = \mathbf{P}(A), \quad \mathbf{P}(B) > 0$$

5 Combinatorics

5.1 k-permutations

The number of ways you can order n things, taken k at a time:

$$\frac{n!}{(n-k)!}$$

5.2 Partitions/Combinations

The number of unique ways to partition a set of n things into subsets of size $n_1, n_2, \ldots, n_m$:

$$\binom{n}{n_1, n_2, \ldots, n_m} = \frac{n!}{n_1! n_2! \cdots n_m!}, \quad \sum_i n_i = n$$

6 Discrete Random Variables

6.1 Common Distributions

Name	X	$p_X(k)$	$\mathbf{E}[X]$	var$[X]$
Bernoulli	1/0	$\begin{cases} p & k=1 \\ 1-p & k=0 \end{cases}$	p	$p(1-p)$
Binomial	hits in n	$\binom{n}{k}p^k(1-p)^{n-k}$	np	$np(1-p)$
Poisson	$\lambda \sim np$	$\frac{\lambda^k}{k!}e^{-\lambda}$	λ	λ
Geometric	trials to hit	$(1-p)^{k-1}p$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Uniform	$\{a, \ldots, b\}$		$\frac{a+b}{2}$	$\frac{(b-a)(b-a+2)}{12}$

6.2 Properties

$$\mathbf{E}[g(X,Y)] = \sum_x \sum_y g(x,y)p_{X,Y}(x,y)$$

$$X = \min(X_1, \ldots, X_n) \rightarrow p_X(k) = \mathbf{P}(X_i > k-1) - \mathbf{P}(X_i > k)$$

6.3 Conditioning

$$p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)} = \frac{p_{X,Y}(x,y)}{\sum_x p_{X,Y}(x,y)}$$

Law of Total Expectation:

$$\mathbf{E}[X] = \sum_y p_Y(y)\mathbf{E}[X|Y=y]$$

6.4 Independence

$$p_{X,Y}(x,y) = p_X(x)p_Y(y)$$
$$p_{X+Y}(w) = \sum_x p_X(x)p_Y(w-x)$$

7 General Random Variables

7.1 Properties

$$\mathbf{P}(a \le X \le b) = \int_a^b f_X(x)dx$$
$$F_X(x) = \mathbf{P}(X \le x)$$
$$F_X(x) = \int_{-\infty}^{x} f_X(x')dx'$$
$$f_X(x) = \frac{dF_X(x)}{dx}$$
$$\mathbf{E}[aX+b] = aE[X]+b$$
$$\text{var}[aX+b] = a^2\text{var}[X]$$
$$\mathbf{E}[aX+bY+c] = a\mathbf{E}[X]+b\mathbf{E}[Y]+c$$
$$\text{var}[X] = \mathbf{E}[X^2] - (\mathbf{E}[X])^2$$

7.2 Independence

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) \rightarrow$$
$$f_{X|Y}(x|y) = f_X(x), \quad \forall (x,y) \in \{f_Y(y) > 0\},$$
$$\mathbf{E}[g(X)h(Y)] = \mathbf{E}[g(X)]\mathbf{E}[h(Y)],$$
$$\text{var}[X+Y] = \text{var}[X] + \text{var}[Y].$$
$$f_{X+Y}(w) = \int_{-\infty}^{\infty} dx\, f_X(x)f_Y(w-x)$$

7.3 Derived Distributions

7.3.1 General

$$f_{Y=aX+b}(y) = \frac{1}{|a|}f_X\left(\frac{y-b}{a}\right)$$
$$Y = g(X) \rightarrow F_Y(y) = \int_{x|g(x)\le y} dx\, f_X(x),$$
$$f_Y(y) = \frac{dF_Y(y)}{dy}.$$

7.3.2 Monotonic function $Y = h^{-1}(X),\ X = h(Y)$

$$f_Y(y) = f_X\big(h(y)\big)\left|\frac{d}{dy}h(y)\right|$$

7.4 Bayes' Rule

$$f_{X|Y}(x|y) = \frac{f_{Y|X}(y|x)f_X(x)}{\int_{-\infty}^{\infty} dx'\, f_{Y|X}(y|x')f_X(x')}$$

$$\mathbf{P}(N=n|Y=y) = \frac{f_{Y|N}(y|n)p_N(n)}{\sum_i p_N(i)f_{Y|N}(y|i)}$$

7.5 Common Distributions

Name	$f_X(x)$	$\mathbf{E}[X]$	var$[X]$
Uniform $[a,b]$	$\frac{1}{b-a},\ a \ge x \ge b$	$\frac{1}{2}a+b$	$\frac{1}{12}(b-a)^2$
Exponential(λ)	$\lambda e^{-\lambda x},\ x \ge 0$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Normal(μ,σ^2)	$\frac{1}{\sigma\sqrt{2\pi}}e^{-(x-\mu)^2/2\sigma^2}$	μ	σ^2

8 Random Variable Theorems

Law of Iterated Expectations:

$$\mathbf{E}\big[\mathbf{E}[X|Y]\big] = \mathbf{E}[X]$$

Law of Total Variance:

$$\text{var}[X] = \mathbf{E}\big[\text{var}[X|Y]\big] + \text{var}\big[\mathbf{E}[X|Y]\big]$$

Random sum of IID RVs:

$$Y = X_1 + \cdots + X_N \rightarrow$$
$$\mathbf{E}[Y] = \mathbf{E}[X]\mathbf{E}[N],$$
$$\text{var}[Y] = \mathbf{E}[N]\text{var}[X] + (\mathbf{E}[X])^2\text{var}[N].$$

9 Stochastic Processes

9.1 Bernoulli Process

$$Y_k = G_1(p) + G_2(p) + \cdots + G_k(p) \quad \text{(Time of } k\text{th arrival)}$$
$$\mathbf{E}[Y_k] = \frac{k}{p}$$
$$\text{var}[Y_k] = \frac{k(1-p)}{p^2}$$
$$p_{Y_k}(t) = \binom{t-1}{k-1}p^k(1-p)^{t-k}$$

Merging independent processes with parameters p and q gives new process with parameter $p+q-pq$.

9.2 Poisson Process

Continuous version of Bernoulli process. Time between events determined by exponential distribution. Number of events in given time duration τ described by Poisson distribution with parameter $\lambda\tau \approx np\tau$.

10 Markov Chains

10.1 Definitions

$$p_{ij} = \mathbf{P}(X_{n+1} = j | X_n = i)$$
$$r_{ij} = \mathbf{P}(X_n = j | X_0 = i)$$

Recurrent	state accessible from all states accessible to it
Transient	not recurrent
Recurrent Class	set of all states accessible from a given recurrent state
Periodic Class	recurrent class which can be broken into sets such that all transitions lead from one set to another

10.2 Chapman-Kolmogorov Equation

$$r_{ij}(n) = \sum_k p_{kj}\, r_{ik}(n-1)$$

10.3 Steady-State

If chain has single aperiodic recurrent class, then states will assymptotically approach probability distribution π_j:

$$\lim_{n\to\infty} r_{ij}(n) = \pi_j, \quad \forall i$$
$$\pi_j = \sum_k \pi_k p_{kj}, \quad \sum_k \pi_k = 1$$

10.4 Birth-Death Process

10.4.1 General

$$\pi_i b_i = \pi_{i+1} d_{i+1}$$
$$\pi_i = \pi_0 \frac{b_0 b_1 \cdots b_{i-1}}{d_1 d_2 \cdots d_i}$$

10.4.2 Queueing Theory

$$\rho \equiv \frac{b}{d}$$
$$\pi_i = \begin{cases} \frac{(1-\rho)\rho^i}{1-\rho^{m+1}} & \rho \neq 1 \\ \frac{1}{m+1} & \rho = 1 \end{cases}$$