6.641: Nonlinear Optics Formulæ

1 Physical Constants

$$c = 2.9979 \times 10^{8} \text{ m/s}$$

$$\epsilon_0 = 8.8542 \times 10^{-12} \text{ F/m}$$

$$\mu_0 = 1.2566 \times 10^{-6} \text{ H/m}$$

$$\eta_0 = 376.73\ \Omega$$

$$q = 1.6022 \times 10^{-19} \text{ C}$$

2 Vector Calculus

$$\int_V \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{a}$$

$$\int_S \nabla \times \mathbf{A} \cdot d\mathbf{a} = \oint_C \mathbf{A} \cdot d\ell$$

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}$$

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot (\nabla \times \mathbf{E}) - \mathbf{E} \cdot (\nabla \times \mathbf{H})$$

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0,\ \nabla \times (\nabla \Phi) = 0$$

3 Maxwell's Equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J}$$

$$\nabla \cdot \mathbf{D} = \rho$$

$$\nabla \cdot \mathbf{B} = 0$$

$$-\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \frac{\partial}{\partial t}\left(\frac{1}{2}\epsilon_0 \mathbf{E}^2 + \frac{1}{2}\mu_0 \mathbf{H}^2\right) + \mathbf{E} \cdot \frac{\partial \mathbf{P}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mu_0 \mathbf{M}}{\partial t} + \mathbf{E} \cdot \mathbf{J}$$

$$I = \frac{1}{2}\Re[\mathbf{E} \times \mathbf{H}^*]$$

$$I = \frac{n}{2\eta_0}|E|^2$$

4 Nonlinear Dielectrics

4.1 Nonlinear Succeptibility

$$P \equiv \epsilon_0 \chi^{(1)} E + P^{\mathrm{NL}}$$

$$P_i(\omega_n + \omega_m) = \sum_{jk} \sum_{(nm)} \chi^{(2)}_{ijk}(\omega_n + \omega_m : \omega_n, \omega_m) E_j(\omega_n) E_k(\omega_m)$$

4.2 Tensor Transformation

$$\chi'_{\alpha\beta\gamma} = T_{\alpha i} \chi_{ijk} T^{-1}_{j\beta} T^{-1}_{k\gamma}$$

$$\mathbf{T}(\theta,\phi) = \begin{bmatrix} \cos\phi & -\cos\theta\sin\phi & \sin\theta\sin\phi \\ \sin\phi & \cos\theta\cos\phi & -\sin\theta\cos\phi \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\mathbf{T}^{-1}(\theta,\phi) = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\cos\theta\sin\phi & \cos\theta\cos\phi & \sin\theta \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{bmatrix}$$

4.3 Anisotropic Materials

$$\tan\rho = \frac{\left(\frac{n_o^2}{n_e^2} - 1\right)\tan\theta}{1 + \frac{n_o^2}{n_e^2}\tan^2\theta}$$

$$n_e(\theta) = \left[\frac{\cos^2\theta}{n_o^2} + \frac{\sin^2\theta}{n_e^2}\right]^{-1/2}$$

4.4 kDB Formalism

$$\nu\kappa\mathbf{D} = v_p^2 \mathbf{D}$$

4.5 Contracted Notation

$$d_{ijk} \equiv \frac{1}{2}\chi^{(2)}_{ijk}$$

$$d_{il} \equiv \begin{bmatrix} d_{111} & d_{122} & d_{133} & d_{123} & d_{131} & d_{112} \\ d_{211} & d_{222} & d_{233} & d_{223} & d_{231} & d_{212} \\ d_{311} & d_{322} & d_{333} & d_{323} & d_{331} & d_{312} \end{bmatrix}$$

$$\begin{bmatrix} P_x(2\omega) \\ P_y(2\omega) \\ P_z(2\omega) \end{bmatrix} = 2\mathbf{d} \begin{bmatrix} E_x^2(\omega) \\ E_y^2(\omega) \\ E_z^2(\omega) \\ 2E_y^2(\omega)E_z^2(\omega) \\ 2E_x^2(\omega)E_z^2(\omega) \\ 2E_x^2(\omega)E_y^2(\omega) \end{bmatrix}$$

$$\begin{bmatrix} P_x(\omega_3) \\ P_y(\omega_3) \\ P_z(\omega_3) \end{bmatrix} = 4\mathbf{d} \begin{bmatrix} E_x(\omega_1)E_x(\omega_2) \\ E_y(\omega_1)E_y(\omega_2) \\ E_z(\omega_1)E_z(\omega_2) \\ E_y(\omega_1)E_z(\omega_2) + E_z(\omega_1)E_y(\omega_2) \\ E_x(\omega_1)E_z(\omega_2) + E_z(\omega_1)E_x(\omega_2) \\ E_x(\omega_1)E_y(\omega_2) + E_x(\omega_1)E_y(\omega_2) \end{bmatrix}$$

$$P(2\omega) = 2d_{\mathrm{eff}}E^2(\omega),\ P(\omega_3) = 4d_{\mathrm{eff}}E(\omega_1)E(\omega_2)$$

4.6 Common Processes

$$P^{(2)}(2\omega_1) = \chi^{(2)} E_1^2 \text{ (SHG)}$$
$$P^{(2)}(\omega_1 + \omega_2) = 2\chi^{(2)} E_1 E_2 \text{ (SFG)}$$
$$P^{(2)}(0) = 2\chi^{(2)} (E_1 E_1^* + E_2 E_2^*) \text{ (OR)}$$
$$P^{(3)}(\omega_1) = \chi^{(3)} (3E_1 E_1^* + 6E_2 E_2^* + 6E_3 E_3^*) E_1$$
$$P^{(3)}(3\omega_1) = \chi^{(3)} E_1^3$$
$$P^{(3)}(\omega_1 + \omega_2 + \omega_3) = 6\chi^{(3)} E_1 E_2 E_3$$
$$P^{(3)}(2\omega_1 + \omega_2) = 3\chi^{(3)} E_1^2 E_2$$

4.7 Electrooptic Effect

$$\kappa'_{ij} = \kappa_{ij} + \frac{1}{\epsilon_0} r_{ijk} E_k^c$$

$$S_{ij} = \begin{bmatrix} \frac{1}{n_o^2} + r_{13}E_z & r_{63}E_z & r_{53}E_z \\ r_{63}E_z & \frac{1}{n_o^2} + r_{23}E_z & r_{43}E_z \\ r_{53}E_z & r_{43}E_z & \frac{1}{n_e^2} + r_{33}E_z \end{bmatrix}$$

$$\Delta n \approx n_o^3 r_{63} E_z$$

4.7.1 Example

$$\frac{1}{n_o'^2} = \frac{1}{n_o^2} + r_{13}E_z$$
$$\frac{d1/n^2}{dn} = -\frac{2}{n^3}$$
$$n'_o \approx n_o - \frac{n_o^3}{2} r_{13} E_z$$

4.8 Kerr Nonlinearity

$$\mathbf{D}(\omega) = \epsilon_0 \left[1 + \chi^{(1)}(\omega) + \chi^{(3)}(\omega : \omega, -\omega, \omega) I(\omega)\right] \mathbf{E}(\omega)$$
$$n = \sqrt{n_0^2 + \chi^{(3)} I} \approx n_0 + n_2 I$$
$$n_2 \equiv \frac{\chi^{(3)}}{2n_0}$$

5 Nonlinear Propagation

5.1 Forced Wave Equation

$$\nabla^2 \mathbf{E} - \mu_0 \epsilon_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \mathbf{P}^{\mathrm{NL}}}{\partial t^2}$$

5.2 Slow Envelope

$$\frac{\partial E(z,t)}{\partial z} + \frac{n}{c} \frac{\partial E(z,t)}{\partial t} = \frac{-jc\mu_0\omega}{2n} P^{\mathrm{NL}}(z,t) e^{j(k-k_p)}$$
$$\frac{\partial E(z,t)}{\partial z} = \frac{-jc\mu_0\omega}{2n} P^{\mathrm{NL}}(z,t) e^{j\Delta k}$$

5.3 Manley-Rowe Relations

$$\frac{1}{\hbar\omega_1} \frac{dI(\omega_1)}{dz} = \frac{1}{\hbar\omega_2} \frac{dI(\omega_2)}{dz} = \frac{1}{\hbar\omega_3} \frac{dI(\omega_3)}{dz}$$
$$M_1 = \frac{I_3}{\omega_3} + \frac{I_2}{\omega_2},\ M_2 = \frac{I_3}{\omega_3} + \frac{I_1}{\omega_1},\ M_3 = \frac{I_2}{\omega_2} - \frac{I_1}{\omega_1}$$

5.4 SHG ($\omega_2 = 2\omega_1$)

5.4.1 Non-depleted Solution

$$E(\omega_2, z = \ell) = \frac{-j\omega_2}{n_2 c} d_{\mathrm{eff}} \ell \left[\frac{\sin \Delta k\ell/2}{\Delta k\ell/2}\right] e^{j\Delta k\ell/2}$$
$$I(\omega_2, \ell) = \frac{2}{n_{\omega_2} c\epsilon_0} \left(\frac{\omega d_{\mathrm{eff}}}{n_\omega c^2}\right)^2 \ell^2 I^2(\omega) \left[\frac{\sin \Delta k\ell/2}{\Delta k\ell/2}\right]^2$$
$$\Delta k = \frac{4\pi}{\lambda_1}[n(\omega_2) - n(\omega_1)]$$

5.4.2 Depleted Solution

$$I(\omega_2, z = \ell) = I(\omega, 0) \tanh^2 \left[\frac{E_0 \omega d_{\mathrm{eff}}}{n_\omega c} \ell\right]$$
$$I(\omega, \ell) = I(\omega, 0) \operatorname{sech}^2[\Gamma \ell]$$

5.4.3 Walk-off Angle ρ

$$\tan \rho = \frac{(n_o^\omega)^2}{2} \left[\frac{1}{(n_e^{2\omega})^2} - \frac{1}{(n_o^{2\omega})^2}\right] \sin 2\theta_p \approx \frac{\Delta n}{n} \sin 2\theta_p$$

5.4.4 Angular Bandwidth

$$\delta\theta \approx \frac{\lambda_1}{2\ell \sin 2\theta_p \Delta n(2\omega)}$$

5.4.5 Wavelength Bandwidth

$$\delta\lambda = \frac{\lambda}{2\ell} \left[\frac{1}{2} \frac{dn}{d\lambda}\bigg|_{2\omega} - \frac{dn}{d\lambda}\bigg|_{\omega}\right]^{-1}$$
$$v_g = \frac{c}{n} \left[1 - \frac{\lambda}{n} \frac{dn}{d\lambda}\right]^{-1}$$

5.5 SFG ($\omega_3 = \omega_1 + \omega_2$)

$$\kappa_i \equiv \frac{\omega_i d_{\mathrm{eff}}}{n_i c}$$

5.5.1 Non-depleted Solution

$$I(\omega_3, \ell) = \frac{2\kappa_3^2 n_3}{n_1 n_2 c\epsilon_0} \ell^2 I(\omega_1) I(\omega_2) \left[\frac{\sin \Delta k\ell/2}{\Delta k\ell/2}\right]^2$$

5.5.2 Strong Pump at ω_1

$$I(\omega_3) = (\omega_3/\omega_2) I_0(\omega_2) \sin^2 \gamma z$$
$$I(\omega_2) = I_0(\omega_2) \cos^2 \gamma z$$
$$\gamma \equiv \sqrt{\frac{2\kappa_2\kappa_3 I(\omega_1)}{n_1 c\epsilon_0}}$$

5.6 DFG/OPA ($\omega_1 = \omega_3 - \omega_2$)

5.6.1 Non-depleted Solution

$$I(\omega_1, \ell) = \frac{2\kappa_1^2 n_1}{n_2 n_3 c\epsilon_0} \ell^2 I(\omega_3) I(\omega_2) \left[\frac{\sin \Delta k\ell/2}{\Delta k\ell/2}\right]^2$$

5.6.2 Strong Pump at ω_3

$$I(\omega_1) = (\omega_1/\omega_2) I_0(\omega_2) \sinh^2 \gamma z$$
$$I(\omega_2) = I_0(\omega_2) \cosh^2 \gamma z$$

5.7 FWM $(\omega_4 = \omega_1 + \omega_2 + \omega_3)$

5.7.1 Strong Pump at ω_1, ω_2

$$\frac{\partial E_4}{\partial z} = -j\kappa_4 E_1 E_2 E_3^*,\ \frac{\partial E_3}{\partial z} = -j\kappa_3 E_1 E_2 E_4^*$$
$$E_4 \equiv E_4(z) e^{\pm\gamma z},\ E_3 \equiv E_3(z) e^{\pm\gamma z}$$
$$\gamma^2 = \kappa_3 \kappa_4 \left|E_1\right|^2 \left|E_2\right|^2$$
$$E_4(z) = A \sinh \gamma z,\ E_3(z) = E_3(0) \cosh \gamma z + B \sinh \gamma z$$
$$A = \frac{E_3^*(0)\gamma}{j\kappa_3 E_1 E_2} = -jE_3^*(0)\sqrt{\frac{\omega_4 n_3}{\omega_3 n_4}},\ B = 0$$

5.8 FWM $(\omega_4 + \omega_3 = \omega_1 + \omega_2)$

5.8.1 Strong Pump at ω_1, ω_2

$$\frac{\partial E_4}{\partial z} = -j\kappa_4 E_1 E_2 E_3,\ \frac{\partial E_3}{\partial z} = -j\kappa_3 E_1 E_2 E_4$$
$$E_4 \equiv E_4(z) e^{\pm j\gamma z},\ E_3 \equiv E_3(z) e^{\pm j\gamma z}$$
$$\gamma^2 = \kappa_3 \kappa_4 E_1^2 E_2^2$$
$$E_4(z) = A \sin \gamma z,\ E_3(z) = E_3(0) \cos \gamma z + B \sin \gamma z$$
$$A = \frac{-j\kappa_4 E_1 E_2 E_3(0)}{\gamma} = -jE_3(0)\sqrt{\frac{\omega_4 n_3}{\omega_3 n_4}},\ B = 0$$

6 Ultrafast Optics

$$\tau_{\text{out}} = \tau\sqrt{1 + \left(\frac{\tau_c}{\tau}\right)^4}$$
$$\tau_c \equiv \sqrt{\ell \frac{\partial^2 \beta}{\partial \omega^2}}\Bigg|_{\omega=\omega_0}$$